ESTIMATES FOR EIGENVALUES OF $\mathcal{L}_r$ OPERATOR ON SELF-SHRINKERS

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ABSTRACT. Let $x : M \to \mathbb{R}^N$ be an n-dimensional compact self-shrinker in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper, we study eigenvalues of the operator $\mathcal{L}_r$ on M, where $\mathcal{L}_r$ is defined by

$$\mathcal{L}_r = e^{\frac{|x|^2}{2}}\mathrm{div}(e^{-\frac{|x|^2}{2}}T^r\nabla\cdot)$$

with T^r denoting a positive definite (0,2)-tensor field on M. We obtain "universal" inequalities for eigenvalues of the operator $\mathcal{L}_r$. These inequalities generalize the result of Cheng and Peng in [8]. Furthermore, we also consider the case that equalities occur.

1. Introduction

A self-shrinker is an immersion $x : M \to \mathbb{R}^N$ of a smooth n-dimensional manifold M into the Euclidean space $\mathbb{R}^N$ which satisfies the quasilinear elliptic system:

$$n\mathbf{H} = -x^{\perp}, \tag{1.1}$$

where $\mathbf{H}$ denotes the mean curvature vector field of the immersion and $\perp$ is the projection onto the normal bundle of M. Self-shrinkers play an important role in the study of the mean curvature flow since they not only correspond to solutions of the mean curvature flow, but also describe all possible blow ups at a given singularity of the mean curvature flow. For more information on self-shrinkers and singularities of mean curvature flow, we refer the readers to [13, 17, 18, 20] and references therein.

In [13], Colding and Minicozzi introduced the following differential operator $\mathcal{L}$ and used it to study self-shrinkers:

$$\mathcal{L} = \Delta - \langle x, \nabla\cdot\rangle, \tag{1.2}$$

where Δ, ∇ denote the Laplacian, the gradient operator on the self-shrinker, respectively, $\langle , \rangle$ stands for the standard inner product in $\mathbb{R}^N$. It is easy to see that the operator can be written as

$$\mathcal{L} = e^{\frac{|x|^2}{2}}\mathrm{div}(e^{-\frac{|x|^2}{2}}\nabla\cdot), \tag{1.3}$$

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where div is the divergent operator on the self-shrinker. Obviously, for a compact self-shrinker M^n, the operator $\mathcal{L}$ is self-adjoint with respect to the measure $e^{-\frac{|x|^2}{2}}dv$. That is,

$$\int\limits_M u\mathcal{L}v\, e^{-\frac{|x|^2}{2}}dv = \int\limits_M v\mathcal{L}u\, e^{-\frac{|x|^2}{2}}dv = -\int\limits_M \langle\nabla u, \nabla v\rangle\, e^{-\frac{|x|^2}{2}}dv \tag{1.4}$$

holds for any $u|_{\partial M} = v|_{\partial M} = 0$. Let T be a positive definite (0,2)-tensor field on M and $f \in C^1(M)$. The following elliptic operator in divergence form

$$\mathcal{L}^{(f,T)} = e^f \mathrm{div}(e^{-f}T\nabla\cdot) \tag{1.5}$$

is very interesting. For any two smooth functions u, v on M with $u|_{\partial M} = v|_{\partial M} = 0$, by the stokes formula, we have

$$\int\limits_M u\mathcal{L}^{(f,T)}v\, d\mu = \int\limits_M v\mathcal{L}^{(f,T)}u\, d\mu = -\int\limits_M \langle\nabla u, T\nabla v\rangle\, d\mu, \tag{1.6}$$

where $d\mu = e^{-f}dv$. That is to say, the operator $\mathcal{L}^{(f,T)}$ is self-adjoint on the space of smooth functions on M vanishing on ∂M with respect to the L^2 inner product under the measure $d\mu = e^{-f}dv$. Therefore, the eigenvalue problem

$$\begin{cases} \mathcal{L}^{(f,T)}(u) = -\lambda u, \text{ in } M, \\ u|_{\partial M} = 0, \end{cases} \tag{1.7}$$

has a real and discrete spectrum:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to +\infty.$$

We put $\lambda_0 = 0$ if $\partial M = \emptyset$. Here each eigenvalue is repeated according to its multiplicity.

In particular, when T is an identity map I, the operator $\mathcal{L}^{(f,T)}$ becomes the the drifting Laplacian $\Delta_f = \Delta - \langle\nabla f, \nabla\rangle$, many interesting estimates have been obtained (cf. [21]); when T is an identity map and $f = 0$, then $\mathcal{L}^{(f,T)}$ becomes the Laplace operator. Let A be the shape operator of an immersion $x : M \to \mathbb{R}^{n+1}(c)$ of an n-dimensional hypersurface M into an $(n+1)$-dimensional space form $\mathbb{R}^{n+1}(c)$ of constant sectional curvature c. Recall that using the characteristic polynomial of A, we can define the elementary symmetric function S_r as follows

$$\det(tI - A) = \sum_{r=0}^{n}(-1)^r S_r t^{n-r}, \tag{1.8}$$

where rth mean curvature H_r is defined by $S_r = \binom{n}{r}H_r$. Then H_1 is the mean curvature H and $H_0 = 1$. The classical Newton transformation T^r are inductively defined by

$$\begin{aligned} T^0 =& I, \\ T^r =& S_r I - T^{r-1}A. \end{aligned} \tag{1.9}$$

For each T^r defined by (1.9), we have a second order differential operator L_r defined by

$$L_r = \operatorname{div}(T^r \nabla \cdot). \tag{1.10}$$

Clearly, the L_r operator can be seen as a special case of $\mathcal{L}^{(f,T)}$ by substituting T and f by T^r and 0, respectively. In particular, $L_0 = \Delta$, L_1 becomes the operator $\Box$ introduced by Cheng-Yau in [6]. Estimates on eigenvalues of L_r operator were studied by many mathematicians. For example, in [1], Alencar, do Carmo and Rosenberg derived the upper bound of the first eigenvalue of the operator L_r on compact hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$:

$$\lambda_1 \int\limits_M H_r \, dv \le c(r) \int\limits_M H_{r+1}^2 \, dv \tag{1.11}$$

and equality holds if and only if M is a sphere, where $c(r) = (n-r)\binom{n}{r}$. For the first eigenvalue of L_r on hypersurfaces of space forms, see [2,3,7,15,16,19] and references therein.

For submanifolds of $\mathbb{R}^N$, we let $\{e_A\}_{A=1}^N$ be an orthonormal basis along M such that $\{e_i\}_{i=1}^n$ are tangent to M and $\{e_\alpha\}_{\alpha=n+1}^N$ are normal to M. Then $A_{ij} = \sum_{\alpha=n+1}^N h_{ij}^\alpha e_\alpha$. If $r \in \{0, 1, \cdots, n-1\}$ is even, for any smooth function u, the operator L_r is defined by (see [4])

$$L_r(u) = \operatorname{div}(T^r \nabla u) = \sum_{i,j} T_{ij}^r u_{ij}$$

since T^r is symmetric and divergence free. Here T^r is given by

$$T_{ij}^r = \frac{1}{r!} \sum_{\substack{i_1 \cdots i_r \\ j_1 \cdots j_r}} \delta_{i_1 \cdots i_r i}^{j_1 \cdots j_r j} \langle A_{i_1 j_1}, A_{i_2 j_2} \rangle \cdots \langle A_{i_{r-1} j_{r-1}}, A_{i_r j_r} \rangle; \tag{1.12}$$

$\delta_{i_1 \cdots i_r i}^{j_1 \cdots j_r j}$ is the generalized Kronecker symbol. By substituting T and f by T^r and $\frac{|x|^2}{2}$, respectively, the operator $\mathcal{L}^{(f,T)}$ becomes

$$\mathcal{L}_r := \mathcal{L}^{(\frac{|x|^2}{2}, T^r)} = L_r - \langle x, T^r \nabla \cdot \rangle, \tag{1.13}$$

which is important in the study of self-shrinkers. For example, when $r = 0$, $\mathcal{L}_0$ becomes the $\mathcal{L}$ operator on the self-shrinker given by (1.2).

In this paper, we assume that T^r is positive definite on M, for some even integer $r \in \{0, 1, \cdots, n-1\}$, namely the operator $\mathcal{L}_r$ is elliptic. Denote by λ_i the i-th eigenvalue of the following eigenvalue problem:

$$\begin{cases} \mathcal{L}_r(u) = -\lambda u, \text{ in } M; \\ u|_{\partial M} = 0, \end{cases} \tag{1.14}$$

and let u_i be the normalized eigenfunction corresponding to λ_i such that $\{u_i\}_1^\infty$ becomes an orthonormal basis of $L^2(M)$ under the weighted measure

$d\mu = e^{-\frac{|x|^2}{2}}dv$, that is

$$\begin{cases} \mathcal{L}_r(u_i) = -\lambda_i u_i, \text{ in } M; \\ u_i|_{\partial M} = 0; \\ \int_M u_i u_j \, d\mu = \delta_{ij}. \end{cases} \tag{1.15}$$

The purpose of this paper is to study eigenvalues of the operator $\mathcal{L}_r$ on a compact self-shrinker M of $\mathbb{R}^N$. We proved the following

Theorem 1.1. *Let $x : M \to \mathbb{R}^N$ be an n-dimensional compact self-shrinker in $\mathbb{R}^N$ with smooth boundary ∂M. Assume that T^r is positive definite on M, for some even integer $r \in \{0, 1, \cdots, n-1\}$. Denote by ξ a positive lower bound of T^r, then the eigenvalues λ_i of the eigenvalue problem* (1.14) *satisfy*

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4(n-r)}{n^2} \max_M(S_r) \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \left(\frac{\lambda_i}{\xi} + \frac{2n - \min\limits_M |x|^2}{4} \right) \tag{1.16}$$

and

$$\sum_{i=1}^{n} \sqrt{\lambda_{i+1} - \lambda_1} \leq 2 \sqrt{(n-r) \max_M(S_r) \left(\frac{\lambda_1}{\xi} + \frac{2n - \min\limits_M |x|^2}{4} \right)}, \tag{1.17}$$

where ξ denotes S_r denotes the rth mean curvature function of x.

Remark 1.1. In particular, we have $S_0 = 1$. Hence, we obtain Theorem 1.1 of Cheng and Peng in [8] by taking $r = 0$ in (1.18).

When $\partial M = \emptyset$, the closed eigenvalue problem

$$\mathcal{L}_r(u) = -\lambda u \tag{1.18}$$

has the following discrete spectrum:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty.$$

We have the following results:

Theorem 1.2. *Let $x : M \to \mathbb{R}^N$ be an n-dimensional compact self-shrinker in $\mathbb{R}^N$ without boundary. Assume that T^r is positive definite on M, for some even integer $r \in \{0, 1, \cdots, n-1\}$. Then the eigenvalues λ_i of the closed eigenvalue problem* (1.18) *satisfy*

$$\lambda_1 \leq 1; \tag{1.19}$$

$$\sum_{i=1}^{n} \sqrt{\lambda_i} \leq \sqrt{\frac{n(n-r)}{\mathrm{vol}(M)} \int_M S_r \, e^{-\frac{|x|^2}{2}} dv}, \tag{1.20}$$

where $\mathrm{vol}(M) = \int_M e^{-\frac{|x|^2}{2}} dv$. *In particular, the equality in* (1.20) *holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_N$.*

Using the fact

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n,$$

we have

$$\sum_{i=1}^{n} \sqrt{\lambda_i} \ge n\sqrt{\lambda_1},$$

and

$$\sum_{i=1}^{n} \sqrt{\lambda_i} \ge \sqrt{\lambda_n}.$$

Therefore, we obtain the following upper bound of the first eigenvalue λ_1 from (1.20):

Corollary 1.3. *Under the assumption of Theorem 1.2, we have*

$$\lambda_1 \le \frac{n-r}{n\,\mathrm{vol}(M)} \int\limits_M S_r\, e^{-\frac{|x|^2}{2}} dv; \tag{1.21}$$

$$\lambda_n \le \frac{n(n-r)}{\mathrm{vol}(M)} \int\limits_M S_r\, e^{-\frac{|x|^2}{2}} dv. \tag{1.22}$$

The equality in (1.21) *holds if and only if* $\lambda_1 = \lambda_2 = \cdots = \lambda_N$.

2. A general inequality for eigenvalues of the operator $\mathcal{L}^{(f,T)}$

In this section, we prove the following general inequalities for eigenvalues of the elliptic operator $\mathcal{L}^{(f,T)}$ defined by (1.5) in divergence form with a weight on compact Riemannian manifolds.

Theorem 2.1. *Let λ_i be the i-th eigenvalue of the problem* (1.7) *and let u_i be the normalized eigenfunction corresponding to λ_i such that $\{u_i\}_1^\infty$ becomes an orthonormal basis of $L^2(M)$ under the weighted measure $d\mu = e^{-f}dv$, that is*

$$\begin{cases} \mathcal{L}^{(f,T)}(u_i) = -\lambda_i u_i, \text{ in } M; \\ u_i|_{\partial M} = 0; \\ \int_M u_i u_j \, d\mu = \delta_{ij}. \end{cases} \tag{2.1}$$

Then for any function $h \in C^2(M)$ and any positive integer k, we have

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int\limits_M u_i^2 \langle \nabla h, T\nabla h\rangle \, d\mu \\ \le &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int\limits_M \Big(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h\rangle \Big)^2 \, d\mu; \end{aligned} \tag{2.2}$$

and

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int\limits_M u_i^2|\nabla h|^2\,d\mu \le \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int\limits_M u_i^2\langle\nabla h, T\nabla h\rangle\,d\mu$$
$$+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int\limits_M\left(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\right)^2 d\mu, \tag{2.3}$$

where δ is a positive constant. Furthermore, if there exists a function $h_A \in C^2(M)$ satisfying

$$\int\limits_M h_A u_1 u_B\,d\mu = 0, \quad \text{for } B=2,\cdots,A, \tag{2.4}$$

then we get

$$(\lambda_{A+1}-\lambda_1)\int\limits_M u_1^2\langle\nabla h_A, T\nabla h_A\rangle\,d\mu \le \int\limits_M[u_1\mathcal{L}^{(f,T)}(h_A)+2\langle\nabla u_1, T\nabla h_A\rangle]^2\,d\mu; \tag{2.5}$$

and

$$\sqrt{\lambda_{A+1}-\lambda_1}\int\limits_M u_1^2|\nabla h_A|^2\,d\mu \le \delta\int\limits_M u_1^2\langle\nabla h_A, T\nabla h_A\rangle\,d\mu$$
$$+\frac{1}{\delta}\int\limits_M\left(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\Delta_f h_A\right)^2 d\mu. \tag{2.6}$$

Remark 2.1. When T is an identity map, the estimate (2.2) becomes the estimate (1.5) in Theorem 1.1 of Xia and Xu [21]; When $f=0$, the estimate (2.3) becomes the estimate (2.4) in Theorem 2.1 of do Carmo, Wang and Xia [14] with $V=0$ and $\rho=1$. For the recent developments about universal inequalities for eigenvalues of the Laplace operator on Riemannian manifolds, we refer to [5, 9–12, 22–28] and the references therein.

Remark 2.2. When $x: M\to\mathbb{R}^N$ is a compact self-shrinker, choosing $T=I$ and $f=\frac{|x|^2}{2}$ in (2.2) and (2.5), respectively, we obtain

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2|\nabla h|^2 d\mu$$
$$\le\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int\limits_M[u_i\mathcal{L}(h)+2\langle\nabla u_i,\nabla h\rangle]^2 d\mu; \tag{2.7}$$

and

$$(\lambda_{A+1}-\lambda_1)\int\limits_M u_1^2\langle\nabla h_A,\nabla h_A\rangle\,d\mu \le \int\limits_M[u_1\mathcal{L}(h_A)+2\langle\nabla u_1,\nabla h_A\rangle]^2\,d\mu. \tag{2.8}$$

Replacing h and h_A by x_A in (2.7) and (2.8) and summing over A, we derive Theorem 1.1 and Proposition 5.1 of Cheng and Peng in [8], respectively. Here $x_A, A = 1, \cdots, N$, denote components of the position vector x.

Proof of the estimate (2.2). Let

$$\varphi_i = hu_i - \sum_{j=1}^{k} a_{ij} u_j, \tag{2.9}$$

where

$$a_{ij} = \int_M h u_i u_j \, d\mu = a_{ji}.$$

It is easy to see that

$$\varphi_i|_{\partial M} = 0, \quad \int_M \varphi_i u_j \, d\mu = 0, \quad \text{for } \forall\ i, j = 1, 2, \cdots, k.$$

We have from the Rayleigh-Ritz inequality

$$\lambda_{k+1} \int_M \varphi_i^2 \, d\mu \leq - \int_M \varphi_i \mathcal{L}^{(f,T)}(\varphi_i) \, d\mu. \tag{2.10}$$

Putting

$$\begin{aligned} \mathcal{L}^{(f,T)}(\varphi_i) &= \mathcal{L}^{(f,T)}(hu_i) + \sum_{j=1}^{k} a_{ij} \lambda_j u_j \\ &= -\lambda_i h u_i + u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T \nabla h \rangle + \sum_{j=1}^{k} a_{ij} \lambda_j u_j \end{aligned} \tag{2.11}$$

in (2.10) gives

$$\begin{aligned} &\lambda_{k+1} \int_M \varphi_i^2 \, d\mu \\ \leq\ &\lambda_i \int_M \varphi_i (h u_i) \, d\mu - \int_M \varphi_i \left(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h \rangle \right) \, d\mu \\ =\ &\lambda_i \int_M \varphi_i^2 \, d\mu - \int_M \varphi_i \left(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h \rangle \right) \, d\mu, \end{aligned} \tag{2.12}$$

which shows that

$$(\lambda_{k+1} - \lambda_i) \int_M \varphi_i^2 \, d\mu \leq P_i, \tag{2.13}$$

where

$$\begin{aligned} P_i &= -\int_M \varphi_i \Big(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h\rangle \Big)\, d\mu \\ &= -\int_M h u_i \Big(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h\rangle \Big)\, d\mu + \sum_{j=1}^{k} a_{ij} b_{ij} \end{aligned} \tag{2.14}$$

with

$$b_{ij} = \int_M u_j \Big(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h\rangle \Big)\, d\mu.$$

Noticing

$$\begin{aligned} \lambda_i a_{ij} &= -\int_M \mathcal{L}^{(f,T)}(u_i) h u_j \, d\mu \\ &= -\int_M \mathcal{L}^{(f,T)}(h u_j) u_i \, d\mu \\ &= -\int_M \Big(h\mathcal{L}^{(f,T)}(u_j) + u_j \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_j, T\nabla h\rangle \Big) u_i \, d\mu \\ &= \lambda_j a_{ij} - \int_M \Big(u_j \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_j, T\nabla h\rangle \Big) u_i \, d\mu \\ &= \lambda_j a_{ij} - b_{ji}. \end{aligned}$$

Hence, we have

$$b_{ji} = (\lambda_j - \lambda_i) a_{ij} = -b_{ij}. \tag{2.15}$$

By virtue of the Stokes formula, we have

$$-\int_M h u_i \Big(u_i \mathcal{L}^{(f,T)}(h) + 2\langle \nabla u_i, T\nabla h\rangle \Big)\, d\mu = \int_M u_i^2 \langle \nabla h, T\nabla h\rangle \, d\mu. \tag{2.16}$$

Thus, we have from (2.14)

$$P_i = \int_M u_i^2 \langle \nabla h, T\nabla h\rangle \, d\mu + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2. \tag{2.17}$$

By the Schwarz inequality and (2.13), we infer

$$
\begin{aligned}
&(\lambda_{k+1}-\lambda_i)P_i^2\\
&=(\lambda_{k+1}-\lambda_i)\left(\int_M \varphi_i\Big(u_i\mathcal{L}^{(f,T)}(h)+2\langle\nabla u_i, T\nabla h\rangle-\sum_{j=1}^k b_{ij}u_j\Big)\,d\mu\right)^2\\
&\le(\lambda_{k+1}-\lambda_i)\int_M \varphi_i^2\,d\mu\cdot\left(\int_M\Big(u_i\mathcal{L}^{(f,T)}(h)+2\langle\nabla u_i, T\nabla h\rangle\Big)^2\,d\mu-\sum_{j=1}^k b_{ij}^2\right)\\
&\le P_i\left(\int_M\Big(u_i\mathcal{L}^{(f,T)}(h)+2\langle\nabla u_i, T\nabla h\rangle\Big)^2\,d\mu-\sum_{j=1}^k b_{ij}^2\right),
\end{aligned}
\tag{2.18}
$$

which gives

$$
(\lambda_{k+1}-\lambda_i)P_i\le\int_M\Big(u_i\mathcal{L}^{(f,T)}(h)+2\langle\nabla u_i, T\nabla h\rangle\Big)^2\,d\mu-\sum_{j=1}^k b_{ij}^2. \tag{2.19}
$$

Multiplying (2.19) by $(\lambda_{k+1}-\lambda_i)$ and taking sum on i from 1 to k, we get

$$
\begin{aligned}
\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2P_i\le&-\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)b_{ij}^2\\
&+\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int_M\Big(u_i\mathcal{L}^{(f,T)}(h)+2\langle\nabla u_i, T\nabla h\rangle\Big)^2\,d\mu.
\end{aligned}
\tag{2.20}
$$

Applying the inequality (2.17), $a_{ij}=a_{ji}$ and $b_{ij}=-b_{ji}$ into (2.20) concludes the proof of the estimate (2.2). □

Proof of the estimate (2.3). From (2.13) and (2.17), we have obtained

$$
(\lambda_{k+1}-\lambda_i)\int_M\varphi_i^2\,d\mu\le\int_M u_i^2\langle\nabla h, T\nabla h\rangle\,d\mu+\sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2. \tag{2.21}
$$

Let

$$
c_{ij}=\int_M u_j\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\Big)\,d\mu.
$$

We have from the Stokes formula

$$
c_{ij}+c_{ji}=\int_M\Big(u_iu_j\Delta_f h+\langle\nabla(u_iu_j),\nabla h\rangle\Big)\,d\mu=0
$$

and

$$\begin{aligned}
&-2\int\limits_M \varphi_i\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\Big)\,d\mu\\
=&-2\int\limits_M hu_i\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\Big)\,d\mu+2\sum_{j=1}^k c_{ij}a_{ij}\\
=&\int\limits_M u_i^2|\nabla h|^2\,d\mu+2\sum_{j=1}^k c_{ij}a_{ij}.
\end{aligned}\tag{2.22}$$

Multiplying (2.22) by $(\lambda_{k+1}-\lambda_i)^2$ and using the Schwarz inequality and (2.21), we obtain

$$\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\Big(\int\limits_M u_i^2|\nabla h|^2\,d\mu+2\sum_{j=1}^k c_{ij}a_{ij}\Big)\\
\le&-2(\lambda_{k+1}-\lambda_i)^2\int\limits_M \varphi_i\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle-\sum_{j=1}^k c_{ij}u_j\Big)\,d\mu\\
\le&\,\delta(\lambda_{k+1}-\lambda_i)^3\int\limits_M \varphi_i^2\,d\mu\\
&+\frac{\lambda_{k+1}-\lambda_i}{\delta}\Big(\int\limits_M\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\Big)^2\,d\mu-\sum_{j=1}^k c_{ij}^2\Big)\\
\le&\,\delta(\lambda_{k+1}-\lambda_i)^2\Big(\int\limits_M u_i^2\langle\nabla h,T\nabla h\rangle\,d\mu+\sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2\Big)\\
&+\frac{\lambda_{k+1}-\lambda_i}{\delta}\Big(\int\limits_M\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\Big)^2\,d\mu-\sum_{j=1}^k c_{ij}^2\Big),
\end{aligned}$$

where δ is any positive constant. Summing over i and noticing $a_{ij}=a_{ji}$, $c_{ij}=-c_{ji}$, we conclude

$$\begin{aligned}
\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int\limits_M u_i^2|\nabla h|^2\,d\mu\le\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int\limits_M u_i^2\langle\nabla h,T\nabla h\rangle\,d\mu\\
+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int\limits_M\Big(\frac{1}{2}u_i\Delta_f h+\langle\nabla u_i,\nabla h\rangle\Big)^2\,d\mu.
\end{aligned}$$

We complete the proof of the estimate (2.3). $\square$

Proof of the estimate (2.5). We let $\varphi_A = h_A u_1 - u_1 \int_M h_A u_1^2 \, d\mu$. Then

$$\int_M \varphi_A u_1 \, d\mu = 0. \tag{2.23}$$

It has been shown from (2.4) that

$$\int_M \varphi_A u_B \, d\mu = 0, \quad \text{for } B = 2, \cdots, A. \tag{2.24}$$

Hence, we have from the Rayleigh-Ritz inequality

$$\lambda_{A+1} \int_M \varphi_A^2 \, d\mu \leq - \int_M \varphi_A \mathcal{L}^{(f,T)}(\varphi_A) \, d\mu. \tag{2.25}$$

According to the Stokes formula, a direct calculation yields

$$\begin{aligned}
& - \int_M \varphi_A \mathcal{L}^{(f,T)}(\varphi_A) \, d\mu \\
= & - \int_M \varphi_A \, \mathcal{L}^{(f,T)}(h_A u_1) \, d\mu \\
= & - \int_M \varphi_A \left(-\lambda_1 h_A u_1 + u_1 \mathcal{L}^{(f,T)}(h_A) + 2\langle \nabla u_1, T \nabla h_A \rangle \right) \, d\mu \\
= & \lambda_1 \int_M \varphi_A^2 \, d\mu - \int_M \varphi_A \left(u_1 \mathcal{L}^{(f,T)}(h_A) + 2\langle \nabla u_1, T \nabla h_A \rangle \right) \, d\mu.
\end{aligned} \tag{2.26}$$

Putting (2.26) into the inequality (2.25) gives

$$\begin{aligned}
& (\lambda_{A+1} - \lambda_1) \int_M \varphi_A^2 \, d\mu \\
\leq & - \int_M \varphi_A \left(u_1 \mathcal{L}^{(f,T)}(h_A) + 2\langle \nabla u_1, T \nabla h_A \rangle \right) \, d\mu \\
= & - \int_M h_A u_1 \left(u_1 \mathcal{L}^{(f,T)}(h_A) + 2\langle \nabla u_1, T \nabla h_A \rangle \right) \, d\mu \\
= & \int_M u_1^2 \langle \nabla h_A, T \nabla h_A \rangle) \, d\mu.
\end{aligned} \tag{2.27}$$

We define

$$\omega_A := - \int_M \varphi_A [u_1 \mathcal{L}^{(f,T)}(h_A) + 2\langle \nabla u_1, T \nabla h_A \rangle] \, d\mu = \int_M u_1^2 \langle \nabla h_A, T \nabla h_A \rangle) \, d\mu. \tag{2.28}$$

Then (2.27) gives

$$(\lambda_{A+1}-\lambda_1)\int\limits_M \varphi_A^2\,d\mu \le \omega_A. \tag{2.29}$$

From the Schwarz inequality and (2.29), we obtain

$$\begin{aligned}
&(\lambda_{A+1}-\lambda_1)\omega_A^2\\
&=(\lambda_{A+1}-\lambda_1)\left(\int\limits_M \varphi_A\left(u_1\mathcal{L}^{(f,T)}(h_A)+2\langle\nabla u_1, T\nabla h_A\rangle\right)d\mu\right)^2\\
&\le(\lambda_{A+1}-\lambda_1)\int\limits_M \varphi_A^2\,d\mu\cdot\int\limits_M\left(u_1\mathcal{L}^{(f,T)}(h_A)+2\langle\nabla u_1, T\nabla h_A\rangle\right)^2 d\mu\\
&\le\omega_A\int\limits_M\left(u_1\mathcal{L}^{(f,T)}(h_A)+2\langle\nabla u_1, T\nabla h_A\rangle\right)^2 d\mu,
\end{aligned} \tag{2.30}$$

which gives

$$(\lambda_{A+1}-\lambda_1)\omega_A\le\int\limits_M\left(u_1\mathcal{L}^{(f,T)}(h_A)+2\langle\nabla u_1, T\nabla h_A\rangle\right)^2 d\mu. \tag{2.31}$$

Combining (2.31) with (2.28) yields the estimate (2.5). □

Proof of the estimate (2.6). On the other hand, from the the Stokes formula again, one gets

$$\begin{aligned}
&-\int\limits_M \varphi_A\Big(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\Delta_f h_A\Big)d\mu\\
&=-\int\limits_M h_A u_1\Big(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\Delta_f h_A\Big)d\mu\\
&=\frac{1}{2}\int\limits_M u_1^2|\nabla h_A|^2\,d\mu.
\end{aligned} \tag{2.32}$$

Therefore, for any positive constant δ, we derive from (2.32)

$$\begin{aligned}
&\sqrt{\lambda_{A+1}-\lambda_1}\int\limits_M u_1^2|\nabla h_A|^2\,d\mu\\
&=-2\sqrt{\lambda_{A+1}-\lambda_1}\int\limits_M \varphi_A\Big(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\Delta_f h_A\Big)d\mu\\
&\le\delta(\lambda_{A+1}-\lambda_1)\int\limits_M\varphi_A^2\,d\mu+\frac{1}{\delta}\int\limits_M\Big(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\Delta_f h_A\Big)^2 d\mu\\
&\le\delta\int\limits_M u_1^2\langle\nabla h_A, T\nabla h_A\rangle)\,d\mu+\frac{1}{\delta}\int\limits_M\Big(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\Delta_f h_A\Big)^2 d\mu,
\end{aligned} \tag{2.33}$$

where in the last inequality we used (2.27). Hence, the desired estimate (2.6) is obtained. □

3. Proof of theorems

Proof of Theorem 1.1. When $x : M \to \mathbb{R}^N$ is a compact self-shrinker, substituting T and f by T^r and $\frac{|x|^2}{2}$ in (2.3) and (2.6), respectively, we obtain for any positive constant δ,

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2|\nabla h|^2\, e^{-\frac{|x|^2}{2}}dv\\ \le\ &\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2\langle\nabla h, T^r\nabla h\rangle\, e^{-\frac{|x|^2}{2}}dv\\ &+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_M\Big(\frac{1}{2}u_i\mathcal{L}(h)+\langle\nabla u_i,\nabla h\rangle\Big)^2 e^{-\frac{|x|^2}{2}}dv,\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\sqrt{\lambda_{A+1}-\lambda_1}\int_M u_1^2|\nabla h_A|^2\, e^{-\frac{|x|^2}{2}}dv \le \delta\int_M u_1^2\langle\nabla h_A, T^r\nabla h_A\rangle)\, e^{-\frac{|x|^2}{2}}dv\\ +\frac{1}{\delta}\int_M\Big(\langle\nabla h_A,\nabla u_1\rangle+\frac{1}{2}u_1\mathcal{L}(h_A)\Big)^2 e^{-\frac{|x|^2}{2}}dv.\end{aligned}\tag{3.2}$$

Let $E_1,\cdots,E_N$ be a canonical orthonormal basis of $\mathbb{R}^N$. Then $x_A = \langle E_A, x\rangle$ and

$$\nabla x_A = \langle E_A, e_i\rangle e_i = E_A^\top,\tag{3.3}$$

where $\top$ denotes the tangent projection to M. Therefore,

$$|\nabla x_A|^2 = |E_A^\top|^2 \le |E_A|^2 = 1, \quad \forall\ A;\tag{3.4}$$

$$\sum_A|\nabla x_A|^2 = n;\tag{3.5}$$

$$\sum_A\langle\nabla x_A,\nabla u_i\rangle^2 = |\nabla u_i|^2;\tag{3.6}$$

$$\begin{aligned}\sum_A\langle\nabla x_A, T^r\nabla x_A\rangle &= \sum_A T^r_{ij}\langle E_A, e_i\rangle\langle E_A, e_j\rangle\\ &= T^r_{ij}\langle e_i, e_j\rangle\\ &= \mathrm{trace}(T^r)\\ &= (n-r)S_r;\end{aligned}\tag{3.7}$$

$$\begin{aligned}\mathcal{L}(x_A) =&\Delta(x_A) - \langle x, \nabla x_A\rangle\\ =&\langle n\mathbf{H}, E_A\rangle - \langle x, E_A^\top\rangle\\ =&-\langle x^\perp, E_A\rangle - \langle x^\top, E_A\rangle\\ =&-\langle x, E_A\rangle\\ =&-x_A;\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\sum_A\int_M u_i\mathcal{L}(x_A)\langle\nabla u_i, \nabla x_A\rangle\, e^{-\frac{|x|^2}{2}}dv\\ =&-\sum_A\int_M u_i x_A\langle\nabla u_i, \nabla x_A\rangle\, e^{-\frac{|x|^2}{2}}dv\\ =&\frac{1}{4}\sum_A\int_M u_i^2\mathcal{L}(x_A^2)\, e^{-\frac{|x|^2}{2}}dv\\ =&\frac{1}{2}\int_M u_i^2(n-|x|^2)\, e^{-\frac{|x|^2}{2}}dv.\end{aligned}\tag{3.9}$$

Here in (3.7), we used Lemma 3.3 in [4] which is still valid for self-shrinkers. Taking $h = x_A$ in (3.1) and summing over A, we get

$$\begin{aligned}&n\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\\ \le&(n-r)\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2 S_r\, e^{-\frac{|x|^2}{2}}dv\\ &+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int_M\Big(\frac{1}{4}u_i|x|^2+|\nabla u_i|^2+\frac{1}{2}u_i^2(n-|x|^2)\Big)\, e^{-\frac{|x|^2}{2}}dv\\ \le&(n-r)\max_M(S_r)\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\\ &+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\Big(\frac{\lambda_i}{\xi}+\frac{2n-\min_M|x|^2}{4}\Big).\end{aligned}\tag{3.10}$$

Minimizing the right hand side of (3.10) by taking

$$\delta=\sqrt{\frac{\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\Big(\frac{\lambda_i}{\xi}+\frac{n}{2}-\frac{1}{4}\min_M|x|^2\Big)}{(n-r)\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\max_M(S_r)}}$$

gives

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4(n-r)}{n^2}\max_M(S_r)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big(\frac{\lambda_i}{\xi}+\frac{2n-\min\limits_M|x|^2}{4}\Big). \tag{3.11}$$

On the other hand, according to the orthogonalization of Gram and Schmidt, we get that there exists an orthogonal $N\times N$-matrix $O=(O_A^B)$ such that

$$\sum_C\int_M O_A^C x_C u_1 u_B = \sum_C O_A^C\int_M x_C u_1 u_B = 0, \quad \text{for } B=2,\dots,A. \tag{3.12}$$

Therefore, taking $h_A=\sum_C O_A^C x_C$ in (3.2), and summing over A, we obtain

$$\begin{aligned} &\sum_A\sqrt{\lambda_{A+1}-\lambda_1}\int_M u_1^2|\nabla h_A|^2\, e^{-\frac{|x|^2}{2}}dv \\ &\leq (n-r)\max_M(S_r)\delta+\frac{1}{\delta}\Big(\frac{\lambda_1}{\xi}+\frac{2n-\min\limits_M|x|^2}{4}\Big). \end{aligned} \tag{3.13}$$

Using (3.4), we infer

$$\begin{aligned} &\sum_{A=1}^{N}\sqrt{\lambda_{A+1}-\lambda_1}|\nabla h_A|^2 \\ \geq&\sum_{i=1}^{n}\sqrt{\lambda_{i+1}-\lambda_1}|\nabla h_i|^2+\sqrt{\lambda_{n+1}-\lambda_1}\sum_{\alpha=n+1}^{N}|\nabla h_\alpha|^2 \\ =&\sum_{i=1}^{n}\sqrt{\lambda_{i+1}-\lambda_1}|\nabla h_i|^2+\sqrt{\lambda_{n+1}-\lambda_1}\left(n-\sum_{j=1}^{n}|\nabla h_j|^2\right) \\ =&\sum_{i=1}^{n}\sqrt{\lambda_{i+1}-\lambda_1}|\nabla h_i|^2+\sqrt{\lambda_{n+1}-\lambda_1}\sum_{j=1}^{n}(1-|\nabla h_j|^2) \\ \geq&\sum_{i=1}^{n}\sqrt{\lambda_{i+1}-\lambda_1}|\nabla h_i|^2+\sum_{j=1}^{n}\sqrt{\lambda_{j+1}-\lambda_1}(1-|\nabla h_j|^2) \\ =&\sum_{i=1}^{n}\sqrt{\lambda_{i+1}-\lambda_1}. \end{aligned} \tag{3.14}$$

Applying (3.14) into (3.13) yields

$$\sum_{i=1}^{n}\sqrt{\lambda_{i+1}-\lambda_1}\leq(n-r)\max_M(S_r)\delta+\frac{1}{\delta}\Big(\frac{\lambda_1}{\xi}+\frac{2n-\min\limits_M|x|^2}{4}\Big). \tag{3.15}$$

Minimizing the right hand side of (3.15) by taking

$$\delta=\sqrt{\frac{\frac{\lambda_1}{\xi}+\frac{2n-\min\limits_M|x|^2}{4}}{(n-r)\max\limits_M(S_r)}}$$

gives the estimate (1.17). We complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. The estimate (1.19) follows from (3.8) directly. We let

$$\varphi_A=h_Au_0-u_0\int_M h_Au_0^2\,e^{-\frac{|x|^2}{2}}dv,$$

where u_0 is the eigenfunction corresponding to $\lambda_0=0$ satisfying

$$\int\limits_M u_0^2\,e^{-\frac{|x|^2}{2}}dv=u_0^2\,\mathrm{vol}(M)=1.$$

Following the proof of the estimate (2.6), we derive the following result by replacing u_1 with u_0 in (2.6) of Theorem 2.1:

Theorem 3.1. *Let λ_i be the i-th eigenvalue of the closed eigenvalue problem* (1.18) *and u_i the normalized eigenfunction corresponding to λ_i such that $\{u_i\}_0^\infty$ becomes an orthonormal basis of $L^2(M)$ under the weighted measure $e^{-\frac{|x|^2}{2}}dv$, that is*

$$\begin{cases}\mathcal{L}_r(u_i)=-\lambda_iu_i,\ \text{in } M;\\ \int\limits_M u_iu_j\,e^{-\frac{|x|^2}{2}}dv=\delta_{ij}.\end{cases}\tag{3.16}$$

If there exists a function $h_A\in C^2(M)$ satisfying

$$\int\limits_M h_Au_0u_B\,e^{-\frac{|x|^2}{2}}dv=0,\quad \text{for } B=1,\cdots,A-1,\tag{3.17}$$

then we have, for any positive constant δ,

$$\begin{aligned}\sqrt{\lambda_A}\int\limits_M|\nabla h_A|^2\,e^{-\frac{|x|^2}{2}}dv\leq&\,\delta\int\limits_M\langle\nabla h_A,T^r\nabla h_A\rangle)\,e^{-\frac{|x|^2}{2}}dv\\&+\frac{1}{4\delta}\int\limits_M(\mathcal{L}(h_A))^2\,e^{-\frac{|x|^2}{2}}dv.\end{aligned}\tag{3.18}$$

According to the orthogonalization of Gram and Schmidt, we get that there exists an orthogonal matrix $O=(O_A^B)$ such that

$$\sum_C\int\limits_M O_A^Cx_Cu_0u_B\,e^{-\frac{|x|^2}{2}}dv=\sum_\gamma O_A^C\int\limits_M x_Cu_0u_B\,e^{-\frac{|x|^2}{2}}dv=0,\tag{3.19}$$

where $B = 1, \dots, A-1$. Taking $h_A = \sum_C O_A^C x_C$ in (3.18), and summing over A from 1 to N, we obtain

$$\begin{aligned}\sum_A \sqrt{\lambda_A} \int\limits_M |\nabla x_A|^2 \, e^{-\frac{|x|^2}{2}} dv \leq & \delta \sum_A \int\limits_M \langle \nabla x_A, T^r \nabla x_A \rangle) \, e^{-\frac{|x|^2}{2}} dv \\ & + \frac{1}{4\delta} \sum_A \int\limits_M (\mathcal{L}(x_A))^2 \, e^{-\frac{|x|^2}{2}} dv.\end{aligned} \tag{3.20}$$

Using (3.8), we have

$$\begin{aligned}\sum_A \int\limits_M (\mathcal{L}(x_A))^2 \, e^{-\frac{|x|^2}{2}} dv &= -\sum_A \int\limits_M x_A \mathcal{L}(x_A) \, e^{-\frac{|x|^2}{2}} dv \\ &= \sum_A \int\limits_M |\nabla x_A|^2 \, e^{-\frac{|x|^2}{2}} dv = n \operatorname{vol}(M).\end{aligned}$$

From the similar argument as in (3.14), we infer

$$\sum_A \sqrt{\lambda_A} |\nabla x_A|^2 \geq \sum_{i=1}^n \sqrt{\lambda_i}. \tag{3.21}$$

Therefore, we derive from (3.20)

$$\sum_{i=1}^n \sqrt{\lambda_i} \operatorname{vol}(M) \leq \delta(n-r) \int\limits_M S_r \, e^{-\frac{|x|^2}{2}} dv + \frac{n \operatorname{vol}(M)}{4\delta}. \tag{3.22}$$

Minimizing the right hand side of (3.22), we derive the desired (1.20). We complete the proof of (1.20).

If the equality in (1.20) occurs, then inequalities (2.25), (2.33) and (3.21) become equalities. Hence, we have

$$\lambda_1 = \lambda_2 = \cdots = \lambda_N; \tag{3.23}$$

$$\mathcal{L}(\varphi_A) = -2\delta \sqrt{\lambda_1} \, \varphi_A \tag{3.24}$$

with $2\delta\sqrt{\lambda_1} = 1$. We remark that the relationship (3.24) is equivalent to (3.8) for compact self-shrinkers. In fact, applying

$$\varphi_A = h_A u_0 - u_0 \int\limits_M h_A u_0^2 \, e^{-\frac{|x|^2}{2}} dv$$

into

$$\mathcal{L}(\varphi_A) = -2\delta \sqrt{\lambda_1} \, \varphi_A,$$

we infer

$$\Big(2\delta\sqrt{\lambda_1} - 1\Big) h_A = 2\delta \sqrt{\lambda_1} u_0^2 \int\limits_M h_A \, e^{-\frac{|x|^2}{2}} dv.$$

It follows from (3.8) that

$$\int_M h_A\, e^{-\frac{|x|^2}{2}} dv = 0.$$

Thus, we obtain $2\delta\sqrt{\lambda_1^{\mathcal{L}_r}} = 1$. $\square$

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